"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

"

ON PROBABILITY MEASURE AND INTEGRALS IN CERTAIN ABSTRACT SPACES

A THESIS

Presented to
the Faculty of the Graduate Division
by
James Joseph Goode

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Applied Mathematics

Georgia Institute of Technology
June, 1958

12T

ON PROBABILITY MEASURE AND INTEGRALS

IN CERTAIN ABSTRACT SPACES

Approved by:

[illegible]

[illegible]

[illegible]

Date of approval:

May 30, 1958

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In 1906 A. Einstein [1] gave a mathematical model for the process of Brownian movement. The mathematical aspects of this model were developed further by M. von Smoluchowski [2] in 1918. To indicate the nature of the Einstein-Smoluchowski model, we shall confine attention to the case of a one-dimensional Brownian movement. Suppose that $x(t)$ denotes the coordinate of a particle at time $t \geq 0$, and suppose that $x(0)=0$. Consider a finite sequence $\{t_k\}$ of t values, with $0 = t_0 < t_1 < t_2 < \cdots < t_n$, and, corresponding to t_k an interval (a_k, b_k), $k = 1,\ldots n$. To the event

$$(1) \qquad a_1 < x(t_1) < b_1,\ a_2 < x(t_2) < b_2,\ \ldots,\ a_n < x(t_n) < b_n$$

is assigned probability measure

$$(2) \qquad \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} \prod_{i=1}^{n} \frac{e^{-\frac{(y_i - y_{i-1})^2}{t_i - t_{i-1}}}}{\sqrt{\pi(t_i - t_{i-1})}} \prod_{i=1}^{n} dy_i$$

where $t_0 = y_0 = 0$. It will be recognized that what is involved here is a process in which the successive increments $x(t_1)-x(0)$, $x(t_2)-x(t_1)$, ..., $x(t_n)-x(t_{n-1})$ are independent random variables, each normally distributed with mean 0 and variance dependent on the time-lag involved. In the above formulation, units are so chosen that the variance of $x(t_k) - x(t_{k-1})$ is $\frac{1}{2}(t_k-t_{k-1})$, $k = 1, 2, \ldots, n$. Indeed, such processes had been discussed as early as 1900 by L. Bachelier [3], though not on a rigorous basis.

The event specified by (1) delimits a set of functions x in the space of real-valued functions on $[0, 1]$, with $x(0)=0$. Such sets, each specified by a finite sequence $\{t_k\}$ of t values with $0 < t_1 < t_2 < \ldots < t_n$, and corresponding one-dimensional intervals $\{(a_k, b_k)\}$, will be called quasi-intervals, following the terminology used by N. Wiener [4]. It is usual, now, to refer to (2) as the Wiener measure of the quasi-interval (1). We shall discuss Wiener's important work on Wiener measure and the Wiener integral later. Since most work in stochastic processes today is based on Kolmogorov's axiomatization [5], it will perhaps be of interest to formulate the present process in that setting.

To recall the Kolmogorov formulation, we note that it concerns an abstract space (sample space) Ω, a σ-field $\mathcal{F}$ of subsets of Ω, where $\mathcal{F}$ includes the empty set $\emptyset$ and the space Ω, and a set function p with domain $\mathcal{F}$ with the properties

(i) $p(\emptyset) = 0,\ p(\Omega) = 1$

(ii) $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F},\ A_iA_j = \emptyset\ (i \neq j)$

$$\Rightarrow p\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1} p(A_j).$$

(iii) $B \subset A,\ A \in \mathcal{F},\ p(A)=0 \Rightarrow B \in \mathcal{F},\ p(B)=0.$

Returning to the Brownian movement process, let us require that all quasi-intervals belong to the possible σ-field $\mathcal{F}$. After verifying certain consistency conditions, which follow from the identity

$$(3) \qquad \frac{e^{-\frac{(y-x)^2}{t}}}{\sqrt{\pi t}} = \int_{-\infty}^{\infty} \frac{e^{-\frac{(z-x)^2}{s}}}{\sqrt{\pi s}} \cdot \frac{e^{-\frac{(y-z)^2}{t-s}}}{\sqrt{\pi(t-s)}}\, dz$$

valid for any s such that $0 < s < t$, an important general theorem due to Kolmogorov may be applied to show the existence of a σ-field $\mathcal{F}$, and probability measure p, on the space Ω of real-valued functions x (with $x(0)=0$), satisfying the Kolmogorov axioms. The field $\mathcal{F}$ contains all quasi-intervals, and for any quasi-interval (1) p prescribes the measure (2). However, J. L. Doob has proved that the set of all continuous functions belonging to Ω , as well as many other kinds of sets, are then not measurable [6]. Doob showed that if one <u>restricts</u> the space Ω to that consisting of <u>continuous</u> functions x (vanishing, as usual, at 0) while retaining the Wiener measure (2) for quasi-intervals (1), then there is a measure p and σ-field $\mathfrak{F}$ of Ω subsets (containing all quasi-intervals) such that the Kolmogorov axioms apply and p yields measure (2) for the quasi-interval (1) and the corresponding measure for any other quasi-interval. In the language of measure theory, p measure is extended from the class of quasi-intervals to the σ-field generated by the quasi-intervals. The measure p can be completed, as usual, by adjoining to $\mathfrak{F}$, all subsets of measurable sets of p-measure zero. For most purposes, Ω is defined to be the set of continuous functions x, with x(0)=0.

In the years from 1923-1930, N. Wiener introduced Wiener measure in a less abstract manner and published several versions of the construction used, as well as numerous fundamental applications of the resulting Wiener integral (which is simply an abstract Lebesgue integral). Assuming the Wiener measure is available, there is a measure space $(\Omega, \mathfrak{F}, p)$. The notion of measurable function is now available : f is a real-valued measurable function on Ω in case $\{x: f(x) \in B\}$ is an $\mathfrak{F}$ set (i.e., belongs to the σ-field $\mathfrak{F}$) for every linear Borel set B. Sometimes f is called a measurable functional to emphasize the fact that the argument x is a function.

CHAPTER II

WIENER INTEGRALS IN A SPACE OF CONTINUOUS FUNCTIONS

In this chapter we will show that it is possible to map the space Ω of all continuous functions vanishing at the origin into a line AB of unit length in such a way that measure is preserved. This will be done in three parts. First, we will define the concept of quasi-intervals and show how they can be mapped into the line AB. Second, we will show that the functions which fail to obey a certain Hölder condition have small measure. In the third part, we shall show how this equi-continuity condition enables us to define the mapping of functions in Ω into points in AB.

The quasi-intervals.—A partition P of $[0,1]$ is a finte set of points, say $P = \{t_0, t_1, \ldots, t_r\}$, such that $0=t_0<t_1<t_2<\ldots<t_{r-1}<t_r=1$. For our purposes, we restrict attention to partitions P with $r=2^n$ (for some positive integer n) and with $t_h=h2^{-n}$, $h=0,1,\ldots,2^n$. Thus the partition points are equally spaced on $[0,1]$. Associated with each number t_h we consider a set of one-dimensional intervals

$[a_{k_h}, b_{k_h}]$, where $a_{k_h} = \tan(k_h\pi 2^{-n})$ and $b_{k_h} = \tan((k_h+1)\pi 2^{-n})$, $-2^{n-1} \leq k_h \leq 2^{n-1}-1$. For $k_h = -2^{n-1}$ we consider an interval of the form $(-\infty, -\cot\frac{\pi}{2^n}]$, and for $k_h = 2^{n-1}-1$ one of the form $[\cot\frac{\pi}{2^n}, +\infty)$. For any fixed t_h the union of the abutting one-dimensional intervals so constructed is clearly $(-\infty, \infty)$, the (one-dimensional) Euclidean space. Consider a set $\{x(t)\}$ of real-valued functions defined on $[0,1]$, and vanishing at 0. Now let $I_n(k_1, k_2, \ldots, k_{2^n})$ be the collection of functions $x(t) \in \Omega$ such that $a_{k_h} \leq x(t_h) \leq b_{k_h}$ for $h = 1, 2, \ldots, 2^n$. Then we call $I_n(k_1, \ldots, k_{2^n})$ a <u>quasi-interval</u>, and we call the corresponding intervals $[a_{k_h}, b_{k_h}]$ $h=1,\ldots,2^n$ the <u>components</u> of $I_n(k_1,\ldots,k_{2^n})$. Let $\mathcal{J}_n$ be the set of the $(2^n)^{2^n}$ quasi-intervals with 2^n components.

Two quasi-intervals belonging to $\mathcal{J}_n$ are said to be <u>disjoint</u> in case one of their component intervals is different, that is $k_h \neq k_h'$ for some h.

Each of the quasi-intervals in $\mathcal{J}_n$ can be decomposed into a finite number of disjoint quasi-intervals of $\mathcal{J}_{n+1}$. In fact a quasi-interval in $\mathcal{J}_n$ can be decomposed into $2^n \cdot 2^{n+1} + 2^n \cdot 2 = 2^{n+1}(2^n+1)$ quasi-intervals since we introduce 2^n new t-points each with its associated 2^{n+1} one-dimensional intervals and we have two possible one-dimensional intervals at the 2^n t-points associated with I_n.

Now let us assume that the functions $\{x(t)\}$ that we shall consider are continuous in $[0,1]$. Let us note this set by Ω. Using the Einstein-Smoluchowski model for Brownian motion discussed in Chapter I, we define the measure $\underline{m}(I_n(k_1,\ldots,k_n))$ of the quasi-interval $I_n(k_1,\ldots,k_{2^n})$ as follows:

$$\underline{m}\,[I_n(k_1,\ldots,k_{2^n})] =$$

$$\int_{a_{k_1}}^{b_{k_1}}\int_{a_{k_2}}^{b_{k_2}}\cdots\int_{a_{k_{2^n}}}^{b_{k_{2^n}}}\prod_{i=1}^{n}\frac{e^{-\frac{(y_i-y_{i-1})^2}{t_i-t_{i-1}}}}{\sqrt{\pi(t_i-t_{i-1})}}\prod_{i=1}^{n}dy_i\,.$$

We notice that, if a fixed quasi-interval $I_n(k_1,\ldots,k_{2^n})$ is decomposed into its $2^{n+1}(2^n+1)$ disjoint quasi-intervals in $\mathcal{I}_{n+1}$, then

$$\underline{m}[I_n(k_1,\ldots,k_{2^n})] = \sum_{j=1}^{2^{n+1}(2^n+1)}\underline{m}[I_{n+1}(k_1^{(j)},\ldots,k_{2^{n+1}}^{(j)})]\,.$$

This follows from the definition of the measure of quasi-intervals and the property of <u>disjointness</u> of the quasi-intervals in $\mathcal{I}_{n+1}$.

Now we consider a mapping which takes the 2^2 quasi-intervals in $\mathcal{I}_1$ into 2^2 disjoint intervals on a unit segment of a straight line AB in such a way that the length of the interval on which a quasi-interval is mapped is equal to the measure of that quasi-interval. In general, the mapping takes each quasi-interval $I_{n+1}(k_1,\ldots,k_{2^{n+1}})$ into an interval on AB of equal measure in such a way that if $I_{n+1}(k_1,\ldots,k_{2^{n+1}})$ is contained in $I_n(l_1,\ldots,l_{2^n})$ then their images stand in the same relation. It is clear that this is possible for all n. We have thus defined a mapping of all of the quasi-intervals into intervals of AB.

<u>The</u> <u>Hölder</u> <u>condition.</u>—We shall now define a mapping of the set of functions $\{x(t)\}$ onto the line AB. In order to define this mapping it will be shown that the set of functions which fail to obey a certain Hölder condition can be enclosed in a denumerable set of quasi-intervals of small total measure.

Let C_h' be the set of all functions in Ω which for fixed h satisfy the inequality

(4) $$|x(t_1)-x(t_2)| > 20h\ |t_1-t_2|^{1/4}$$

for some t_1 and t_2 in $[0,1]$. We will show that this set C_h' can be enclosed in a denumerable union C_h of quasi-intervals such that, for any positive integer m, $\underline{m}(C_h)= o(h^{-m})$ as $h\to\infty$.

To do this, we shall first show that any function satisfying (4) for some t_1 and t_2 also satisfies the inequality.

$$(5) \qquad \left|x\left(\frac{m}{2^i}\right)-x\left(\frac{m+1}{2^i}\right)\right| > h2^{-i/4}$$

for some i and some $m \leq 2^i-1$.

Consider the binary representations of the numbers t_1 and t_2:

$$t_1 = 0.a_1a_2\ldots \quad a_i = 0 \text{ or } 1$$

$$t_2 = 0.b_1b_2\ldots \quad b_i = 0 \text{ or } 1.$$

Suppose, for definiteness, that $t_1 < t_2$. Let h be the first position at which the binary representations of t_1 and t_2 differ, i.e., $a_h=0$ and $b_h=1$. Then let

$$t_{10} = t_{20} = 0.a_1\ldots a_{h-1}1000\ldots.$$

Let j be the number of positions in which t_{20} in the above terminating form agrees with t_2 and with t_1 in a form terminating in ones. Clearly, the, $j \geq h$ and $|t_2-t_1| > 2^{-j-1}$. Now we construct the number t_{11} by considering

$$t_{10} = 0.a_1a_2\ldots a_{h-1}01111\ldots$$

and letting t_{11} agree with t_1 and t_{10} up to the position at which they first disagree and putting a one in this position followed by zeros. In general, we construct $t_{1(i+1)}$ to agree with t_{1i} and t_1 as far as possible (considering t_{1i} as terminating in ones to make the agreement as far as possible) and put a one in the first position of disagreement. We construct $t_{2(i+1)}$

between t_2 and t_{2i} in the same way.

For example, let $t_1=0.0111001000\bar{0}\ldots$ and $t_2=0.100010010101\overline{10}\ldots.$ Then

$$\begin{array}{ll}
t_{10}=0.100\bar{0}\ldots & t_{20}=0.100\bar{0}\ldots \\
t_{11}=0.0111100\bar{0}\ldots & t_{21}=0.1000100\bar{0}\ldots \\
t_{12}=0.01110100\bar{0}\ldots & t_{22}=0.1000100100\bar{0}\ldots \\
t_{13}=0.0111001100\bar{0}\ldots & t_{23}=0.100010010100\bar{0}\ldots \\
t_{14}=0.01110010100\bar{0}\ldots & t_{24}=0.10001001010100\bar{0}\ldots \\
t_{15}=0.011100100100\bar{0}\ldots & t_{25}=0.1000100101010100\bar{0}\ldots \\
\vdots & \vdots
\end{array}$$

Now by the above process we have decomposed the interval (t_1,t_2) into a denumerable collection of sub-intervals with at most two sub-intervals of length 2^{-j-k} for any positive integer k, and no intervals of length greater than 2^{-j-1}. Each sub-interval is of the form $\left[\frac{m}{2^{j+k}}, \frac{m+1}{2^{j+k}}\right]$ for some k and some m.

Since $|t_1-t_2| \geq 2^{-j-1}$ we deduce from condition (4) that for some t_1 and t_2

$$|x(t_1)-x(t_2)| \geq 20h2^{-\frac{j+1}{4}} > 2h\frac{2^{-\frac{j+1}{4}}}{1-2^{-1/4}}$$

since $10 > \frac{1}{1-2^{1/4}}$. Since $\sum_{k=1}^{\infty} 2^{-\frac{j+k}{4}} = \frac{2^{-\frac{j+1}{4}}}{1-2^{-1/4}}$, we have

$$(6) \qquad |x(t_1)-x(t_2)| > 2h\sum_{k=1}^{\infty} 2^{-\frac{j+k}{4}}.$$

Now we note that

$$|x(t_1)-x(t_2)| \leq |x(t_{1n})-x(t_{2n})| + |x(t_1)-x(t_{1n})| + |x(t_{2n})-x(t_2)|$$

for every n. But

$$|x(t_{1n})-x(t_{2n})| = \sum_{i=0}^{n} |x(t_{1i})-x(t_{1(i+1)})| + \sum_{i=0}^{n} |x(t_{2i})-x(t_{2(i+1)})|.$$

Then on considering the limit as $n \to \infty$, and taking into account the continuity of x we note that

$$(7) \qquad |x(t_1)-x(t_2)| \leq$$

$$\sum_{i=0}^{\infty} (|x(t_{1i})-x(t_{1(i+1)})| + |x(t_{2i})-x(t_{2(i+1)})|).$$

Recalling our analysis of the interval (t_1, t_2), and using (6) and (7) we find that there must exist positive integers m and k such that

$$\left|x\left(\frac{m}{2^{j+k}}\right) - x\left(\frac{m+1}{2^{j+k}}\right)\right| > h2^{-\frac{j+k}{4}},$$

for otherwise inequality (6) would not hold. Thus we have

(5) $$\left|x\left(\frac{m}{2^i}\right)-x\left(\frac{m+1}{2^i}\right)\right|>h2^{-i/4}$$

for some m and some i.

Now we calculate the measure of the functions satisfying (5). Let C_h^{mi} be the set of all functions x which satisfy (5) for fixed m, i, and h. Then the measure of the set Q_h^{mi} of quasi-intervals in $\mathcal{J}_h$ which contain functions in C_h^{mi} is

(8) $$\underline{m}(Q_h^{mi}) = \frac{1}{\pi 2^{-i}} \int_{-\infty}^{\infty}\left\{\int_a^{\infty} e^{-\left[\frac{y_1^{\;2}}{2^{-i}}+\frac{(y_2-y_1)^2}{2^{-i}}\right]} dy_2 + \int_{-\infty}^{b} e^{-\left[\frac{y_1^{\;2}}{2^{-i}}+\frac{(y_2-y_1)^2}{2^{-i}}\right]} dy_2\right\} dy_1$$

where a is the lower end-point of the interval at $t=\frac{m+1}{2^i}$ which contains $y_1+h2^{-i/4}$ and b is the upper end-point of the interval at $t=\frac{m+1}{2^i}$ which contains $y_1-h2^{-i/4}$. This follows from the basic assumption of independent increments. Hence we have

$$\underline{m}(Q_h^{mi}) = \frac{1}{\pi 2^{-i}} \int_{-\infty}^{\infty} \Big[\Big[\int_{y_1+h2^{-i/4}}^{\infty} \emptyset(y_1,y_2)dy_2 + \int_{-\infty}^{y_1-h2^{-i/4}} \emptyset(y_1,y_2)dy_2 +$$

$$\int_{a}^{y_1+h2^{-i/4}} \emptyset(y_1,y_2)dy_2 + \int_{y_1-h2^{-i/4}}^{b} \emptyset(y_1,y_2)dy_2 \Big] dy_1$$

where $\emptyset(y_1,y_2)$ is the integrand of equation (8). Letting $z_2 = y_2-y_1$, $z_1 = y_1$ in the first integral and $z_1 = y_1$, $z_2 = y_1-y_2$ in the second, we obtain

$$\underline{m}(Q_h^{mi}) = \frac{1}{\pi 2^{-i}} \int_{-\infty}^{\infty} \Big[2 \int_{h2^{-i/4}}^{\infty} e^{-\frac{z_1^2}{2^{-i}} - \frac{z_2^2}{2^{-i}}} dz_2 \Big] dz_1 + I_3 + I_4$$

$$= \frac{2}{\sqrt{\pi 2^{-i}}} \int_{h2^{-i/4}}^{\infty} e^{-\frac{z_2^2}{2^{-i}}} dz_2 + I_3 + I_4.$$

On letting $y = \dfrac{z_2}{\sqrt{2^{-i}}}$ we obtain

$$\underline{m}(Q_h^{mi}) = \frac{2}{\sqrt{\pi}} \int_{h2^{i/4}}^{\infty} e^{-y^2} dy + I_3 + I_4.$$

Now by choosing n large enough we can make $I_3+I_4 < \frac{\epsilon}{2^{2i}}$ for any given $\epsilon > 0$. Thus the total measure of this set Q_h^{mi} of quasi-intervals in $\mathcal{J}_n$ is, for large enough n, less than

$$\frac{2}{\sqrt{\pi}} \int_{h2^{i/4}}^{\infty} e^{-y^2} dy + \frac{\epsilon}{2^{2i}}.$$

Now summing over all m from 0 to 2^i-1, we find

$$\sum_{0}^{2^i-1} \underline{m}(Q_h^{mi}) < \frac{2^{i+1}}{\sqrt{\pi}} \int_{h2^{i/4}}^{\infty} e^{-y^2} dy + \frac{\epsilon}{2^i}$$

and summing over all $i \geq 1$ we find

$$\underline{m}(C_h) < \epsilon + \sum_{i=1}^{\infty} \frac{2^{i+1}}{\sqrt{\pi}} \int_{h2^{i/4}}^{\infty} e^{-z^2} dz,$$

and, for $h > 1$,

$$\underline{m}(C_h) < \epsilon + \sum_{i=1}^{\infty} \frac{2^{i+1}}{\sqrt{\pi}} \int_{h2^{i/4}}^{\infty} e^{-z} dz$$

$$< \epsilon + \sum_{i=1}^{\infty} \frac{2^{i+1}}{\sqrt{\pi}} e^{-h2^{i/4}}.$$

Now since, for any m, h can be chosen sufficiently large that

$$e^{-h2^{i/4}} < h^{-4m} 2^{-mi},$$

it follows that the total measure of the set C_h of quasi-intervals satisfies the inequality

$$\underline{m}(C_h) < \epsilon + h^{-4m} \sum_{i=1}^{\infty} \frac{2^{-(m-1)i+1}}{\sqrt{\pi}},$$

for $h > H_m$. Hence $\underline{m}(C_h)$ is clearly $o(h^{-m})$ as $h \to \infty$ for any fixed positive integer m.

All functions not in C_h satisfy the Hölder condition

$$(9) \qquad |x(t_1)-x(t_2)| \leq 20h\,|t_1-t_2|^{1/4}$$

for all t_1 and t_2 in $[0,1]$.

<u>The mapping of functions to points</u>.—With each point on AB, except for the end-points of the image intervals, is associated, by the Cantor Theorem on Nested Sets,* a unique sequence of decreasing image intervals. The end-points, however, have zero measure. This sequence, in turn, corresponds to a monotonic sequence of quasi-intervals:

$$(10) \qquad I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots.$$

---

*<u>Cantor's Theorem</u>. In a complete metric space, every decreasing sequence of closed nonempty sets S_n such that the sequence of their diameters $d(S_n)$ has limit zero as $n \to \infty$ has a nonempty intersection consisting of exactly one point.

Either I_n, for large enough n, is contained in C_h, or else, for all n, each quasi-interval I_n contains functions which satisfy (9) for every t_1 and t_2 that are <u>terminating binaries</u>. In the latter case, the lengths $b_{k_h} - a_{k_h} \to 0$ at every terminating binary t_h. If $a_{k_h} \to -\infty$ or $b_{k_h} \to \infty$, the sequence would be one for which I_n is contained in C_h for large enough n. Again using the nested interval theorem, the sequence (10) selects exactly one point at each terminating binary t_h and we shall call this number $x(t_h)$. The function x so defined satisfies condition (9) on the terminating binaries. Now the continuous extension of x to the closure of the set of terminating binaries will satisfy the Hölder condition (9), and is unique [7]. Since the closure of the terminating binaries on $[0,1]$ is $[0,1]$, we have a mapping of points on AB to a set of functions defined on $[0,1]$.

Thus we have a mapping of points on the line AB into functions in Ω. This mapping is not defined on the end-points of the intervals in AB, nor is it defined on the set of points which correspond to the quasi-intervals in C_h. The set of end-points has Lebesgue measure zero. The set corresponding to C_h has measure $o(h^{-m})$. The other points on AB map into a set of equi-continuous functions in Ω.

Now it is clear that to any function satisfying (9) for all t_1 and t_2 in $[0,1]$, there corresponds a decreasing sequence of quasi-intervals which determines exactly one point on AB or a countable set of end-points. But this latter set clearly has measure zero.

Hence, for any fixed integer m, except for a set of functions of measure $o(h^{-m})$ we have a one-to-one mapping of the space Ω (of continuous functions vanishing at the origin) onto the real line interval AB. But since this is true for arbitrary $h > H_m$, we consider a sequence $h_j \to \infty$ and note that the corresponding mappings extend to more functions and points. Hence in the limit the mapping is one-to-one except on a set of measure zero.

This then enables us to define the Wiener integral of a functional $\emptyset(x(\cdot))$. This functional, through the mapping, determines a function on the line AB. A functional is Wiener measurable if its corresponding function is Lebesgue measurable. If $\emptyset$ is Wiener measurable on Ω we write

$$\int_{\Omega} \emptyset(x(\cdot))d_w x$$

as the Wiener integral or expected value of $\emptyset$ over the space Ω of all continuous functions on $[0,1]$ vanishing at the origin.

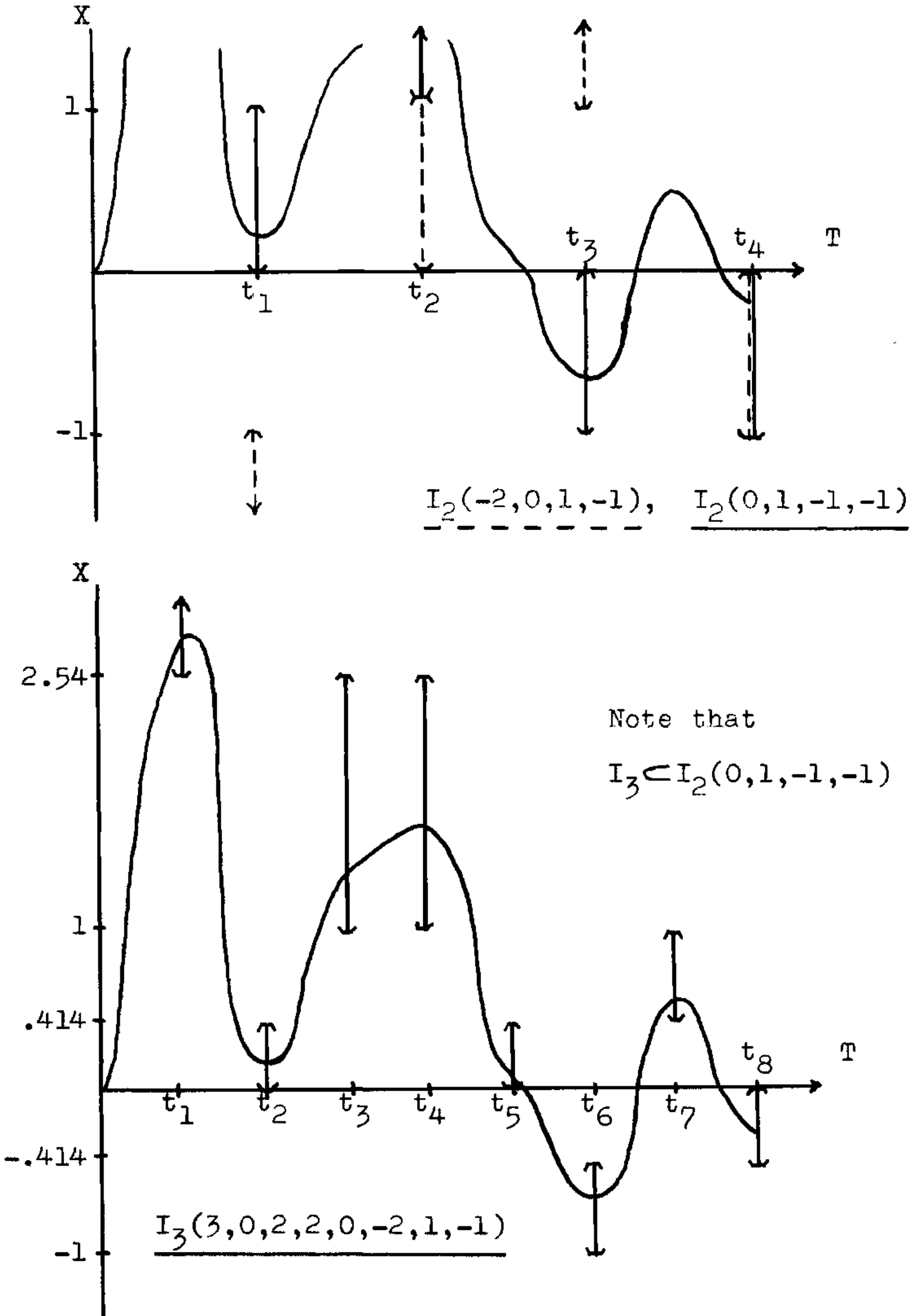

Fig. 1

Sample Quasi-Intervals

CHAPTER III

EXAMPLES OF WIENER INTEGRALS

Expected value of a product $x(t_1)\cdot x(t_2)$.—Suppose we wish to find the average or expected value of the product $x(t_1)\cdot x(t_2)$ for $t_1 < t_2$ over the whole space Ω of continuous functions. From the definition of the measure on the space, we have

$$\int_{\Omega} x(t_1)x(t_2)d_w x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1\cdot y_2 \frac{e^{-\frac{y_1^2}{t_1}-\frac{(y_2-y_1)^2}{t_2-t_1}}}{\pi\sqrt{t_1(t_2-t_1)}}\, dy_2 dy_1 .$$

Letting $y_1 = \sqrt{t_1}\, z_1$ and $y_2 = \sqrt{t_1}\, z_1 + \sqrt{t_2-t_1}\, z_2$, we obtain

$$\int_{\Omega} x(t_1)x(t_2)d_w x = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(t_1 z_1^2 + \sqrt{t_1(t_2-t_1)}\, z_1 z_2)\cdot e^{-z_1^2-z_2^2}\, dz_2 dz_1$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} t_1 z_1^2\, e^{-z_1^2}\int_{-\infty}^{\infty} e^{-z_1^2}\, dz_2 dz_1 +$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{t_1(t_2-t_1)}\, e^{-z_1^2}\int_{-\infty}^{\infty} z_2\, e^{-z_2^2}\, dz_2 dz_1 .$$

Since $\int_{-\infty}^{\infty} xe^{-x^2} dx = 0$, the second term is clearly zero.

Since $\int_{-\infty}^{\infty} e^{-z_2^2}\,dz_2 = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} z_1^2\, e^{-z_1^2}\,dz_1 = \frac{\sqrt{\pi}}{2}$, we obtain

$$\int_{\Omega} x(t_1)x(t_2)d_w x = \frac{t_1}{2}$$

Now $\int_{\Omega} [x(t_1)]^2\, d_w x = \frac{1}{\sqrt{\pi t_1}} \int_{-\infty}^{\infty} y^2\, e^{-\frac{y^2}{t_1}}\, dy = \frac{t_1}{2}$.

Thus

$$\int_{\Omega} x(t_1)x(t_2)d_w x = \frac{t_1}{2} \qquad \text{for } t_1 \leq t_2. \tag{11}$$

<u>Expected value of $[x(t)]^n$</u>.—Now consider the expected value of $[x(t)]^n$.

$$\int_{\Omega} [x(t)]^n\, d_w x = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} y^n\, e^{-\frac{y^2}{t}}\, dy = \frac{t^{n/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^n\, e^{-z^2} dz.$$

Clearly, for n odd we obtain zero. For n even we integrate by parts to obtain

$$\int_{\Omega} [x(t)]^{2n}\, d_w x = \frac{t^n}{\sqrt{\pi}}\, \frac{2n-1}{2} \int_{-\infty}^{\infty} z^{2n-2}\, e^{-z^2} dz$$

Repeating the process we obtain

$$(12) \qquad \int_{\Omega} [x(t)]^n d_w x = \frac{t^n (2n-1)(2n-3)\ldots(3)(1)}{2^n}$$

$$= \frac{t^n (2n)!}{2^{2n} n!}$$

Expected value of cos [x(t)].—Now consider the expected value of $\cos[x(t)]$,

$$\int_{\Omega} \cos[x(t)]\, d_w x = \int_{\Omega} \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-1)^n [x(t)]^{2n}}{2n\ !}\, d_w x.$$

Since the sums are bounded for k large enough, we use Lebesgue's dominated convergence theorem to obtain

$$\int_{\Omega} \cos[x(t)]\, d_w x = \lim_{k\to\infty} \int_{\Omega} \sum^{k} \ldots d_w x =$$

$$= \lim_{k\to\infty} \sum^{k} \int_{\Omega} \ldots d_w x$$

$$= \sum_{n=0}^{\infty} \int_{\Omega} \frac{(-1)^n [x(t)]^{2n}}{(2n)!}\, d_w x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_{\Omega} [x(t)]^n\, d_w x$$

and by (12) we have

$$\int_{\Omega} \cos[x(t)]\, d_w x = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n} n!}.$$

But this sum is simply the MacLaurin expansion for $e^{-\frac{1}{4}t}$.

Therefore

$$\int_{\Omega} \cos[x(t)]\, d_w x = e^{-\frac{1}{4}t} \tag{13}$$

<u>Expected value of $\int_0^1 x(t)dt$</u>.—Now consider finding the expected value of the functional $\int_0^1 x(t)dt$. The functional $F(x(\cdot),t) = x(t)$ is defined on the space $\Omega \times [0,1]$. F is integrable on the product space since

$$\int_{\Omega} \int_0^1 x(t)dt d_w x = \int_{\Omega} \lim_{n\to\infty} \sum_{k=1}^{n} x(\tfrac{k}{n}) \tfrac{1}{n}\, d_w x$$

Note that the partial sums are bounded by $M_x = \max_{t\in[0,1]} x(t)$ which is integrable in Ω. The Lebesgue dominated convergence theorem then guarantees existence of the finite integral. Therefore Fubini's theorem applies and we can interchange order of integration and obtain

$$\int_{\Omega} \int_0^1 x(t)dt d_w x = \int_0^1 \int_{\Omega} x(t) d_w x dt = 0$$

The inner integral is zero.

<u>Expected value of $[\int_0^1 x(t)dt]^2$</u>.—Now consider

$$\int_{\Omega} [\int_0^1 x(t)dt]^2 d_w x.$$

If we write the integrand as

$$\int_0^1 x(t_1)dt_1 \cdot \int_0^1 x(t_2)dt_2 = \int_0^1 \int_0^1 x(t_1)x(t_2)dt_1dt_2.$$

This may be written as

$$\int_0^1 \int_0^{t_2} x(t_1)x(t_2)dt_1dt_2 + \int_0^1 \int_0^{t_1} x(t_1)x(t_2)dt_2dt_1$$

$$= 2\int_0^1 \int_0^{t_2} x(t_1)x(t_2)dt_1dt_2.$$

Using Fubini's theorem as before we write

$$\int_{\Omega} [\int_0^1 x(t)dt]^2 d_w x = 2\int_0^1 \int_0^{t_2} \int_{\Omega} x(t_1)x(t_2)d_w x dt_1dt_2.$$

And since in the region of integration $t_1 \leq t_2$, we have by (11)

$$\int_{\Omega} [\int_0^1 x(t)dt]^2 d_w x = 2\int_0^1 \int_0^{t_2} \frac{t_1}{2}\, dt_1dt_2$$

$$= \int_0^1 \frac{t_2^{\,2}}{2}\, dt_2$$

Thus

(14) $$\int_{\Omega} [\int_0^1 x(t)dt]^2 d_w x = \frac{1}{6}.$$

<u>Expected value of</u> $e^{-\int_0^t V(x(s))ds}$.—Now let us consider a more difficult problem. Consider the problem of evaluating

$$\int_{\Omega} e^{-\int_0^t V(x(s))ds} d_w x$$

where $0 \leq t \leq 1$ and $V(x)$ is continuous. This problem is considered by M. Kac in [8]. We will show how this problem is related to certain problems in theoretical physics. In particular, we will show that the above integral can be evaluated by solving a differential equation of the Schrödinger type. we will also evaluate the integral for $V(x) = x^2$. Assume further that

(15) $$0 \leq V(x) < M$$

Note that

$$e^{-\int_0^t V(x(s))ds} = \sum_{k=0} \frac{(-1)^k}{k!} [\int_0^t V(x(s))ds]^k.$$

Since

$$0 \le \int_0^t V(x(s))ds < M$$

we can, as in obtaining formula (13), write

$$(16) \quad \int_\Omega e^{-\int_0^t V(x(s))ds} d_wx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_\Omega \left[\int_0^t V(x(s))ds\right]^k d_wx.$$

The problem then becomes to calculate

$$u_k = \int_\Omega \left[\int_0^t V(x(s))ds\right]^k d_wx.$$

For $k=1$, since $V(x) < M$, we have by Fubini's theorem

$$\int_\Omega \int_0^t V(x(s))ds d_wx = \int_0^t \int_\Omega V(x(s))d_wx ds$$

$$= \int_0^t \int_{-\infty}^{\infty} V(y) \frac{e^{-\frac{y^2}{s}}}{\sqrt{\pi s}} dyds.$$

For $k=2$ we have, as in obtaining (14),

$$\int_{\Omega} \left[\int_0^t V(x(s))ds\right]^2 d_w x =$$

$$2\int_{\Omega} \left[\int_0^t \int_0^{s_2} V(x(s_1))V(x(s_2))ds_1 ds_2\right] d_w x$$

$$= 2\int_0^t \int_0^{s_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(y_1)V(y_2) \frac{e^{-\sum_{i=1}^{2} \frac{(y_i - y_{i-1})^2}{s_i - s_{i-1}}}}{\sqrt{\pi^2 s_1 (s_2 - s_1)}} dy_1 dy_2 ds_1 ds_2 .$$

In general we have

$$\text{(17)} \qquad u_k = k! \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{k} V(y_i) \frac{e^{-\frac{(y_i - y_{i-1})^2}{s_i - s_{i-1}}}}{\sqrt{\pi \, s_i - s_{i-1}}} \cdot \prod_{i=1}^{k} dy_i \prod_{i=1}^{k} ds_i .$$

We now introduce the functions Q_n defined as follows:

$$\text{(18)} \qquad Q_o(x,t) = \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}}$$

$$(19) \qquad Q_{n+1}(s,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y) Q_n(y,s)\,dy\,ds$$

$$n = 0, 1, \ldots .$$

By changing orders of integration in (17), we see that

$$(20) \qquad u_k = k! \int_{-\infty}^{\infty} Q_k(x,t)$$

From (18) and (19) we have

$$(21) \qquad 0 \leq Q_n(x,t) \leq \frac{(Mt)^n}{n!} Q_o(x,t).$$

Thus we may define

$$(22) \qquad Q(x,t) = \sum_{k=0}^{\infty} (-1)^k Q_k(x,t).$$

From (21) we have

$$(23) \qquad |Q(x,t)| \leq e^{Mt} Q_o(x,t).$$

Now we note that

$$\int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y) Q(y,s)\,dy\,ds =$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y) Q_k(y,s)\,dy\,ds$$

$$= \sum_{k=0} (-1)^k\, Q_{k+1}(x,t) = -Q(x,t) + Q_o(x,t).$$

Therefore Q satisfies the integral equation

$$(24) \qquad Q(x,t) + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y) Q(y,s)\,dy\,ds = Q_o(x,t).$$

Now it follows from (16) and (20) that

$$\int_{\Omega} e^{-\int_0^t V(x(s))ds}\, d_w x = \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{\infty} Q_k(x,t)\,dx.$$

Thus by (22) we have

$$(25) \qquad \int_{\Omega} e^{-\int_0^t V(x(s))ds}\, d_w x = \int_{-\infty}^{\infty} Q(x,t)\,dx.$$

We interchanged summation and integration by (23).

Now in the above we have been dealing with the expected value of

$$e^{-\int_0^t V(x(s))ds}$$

over the whole space Ω. If we restrict ourselves to the subset S_t of Ω, where $S_t = \{x : a \le x(t) \le b\}$, then by minor modification of the above development we have

$$\int_{S_t} e^{-\int_0^t V(x(s))ds} d_w x = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{S_t} \left[\int_0^t V(x(s))ds\right]^k d_w x.$$

We find as before that

$$\int_{S_t} \left[\int_0^t V(x(s))ds\right]^k d_w x = k! \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \int_{S_t} \prod_{i=1}^{k} V(x(s_i)) d_w x \prod_{i=1}^{k} ds_i.$$

But

$$\int_{S_t} \prod V(x(s_i)) d_w x = \int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{k} V(y_i).$$

$$\cdot \frac{e^{-\frac{(x-y_k)^2}{t-s_k}}}{\sqrt{\pi(t-s_k)}} \prod_{i=1}^{k} \frac{e^{-\frac{(y_i-y_{i-1})^2}{(s_i-s_{i-1})}}}{\sqrt{\pi(s_i-s_{i-1})}} \prod_{i=1}^{k} dy_i \, dx.$$

From this we see that

$$\int_{S_t} e^{-\int_0^t V(x(s))ds} d_w x = \int_a^b Q(x,t) dx. \tag{26}$$

It is clear from (26) that $Q(x,t) \geq 0$.

Now let us remove the condition that $V(x) < M$. We do this by setting

$$V_M(x) = \begin{cases} V(x), & \text{if } V(x) = M \\ M, & \text{if } V(x) = M \end{cases}$$

Denote the corresponding Q functions by $Q^{(M)}$. Since the integrand is dominated by one, we use the Lebesgue dominated convergence theorem to obtain

$$\lim_{M \to \infty} \int_{S_t} e^{-\int_0^t V_M(x(s))ds} d_w x = \int_{S_t} e^{-\int_0^t V(x(s))ds} d_w x.$$

Now since

$$e^{-\int_0^t V_M(x(s))ds} \geq e^{-\int_0^t V(x(s))ds},$$

we see from (26) that the functions $\{Q^{(M)}\}$ form a decreasing sequence in M for each point (x,t). Since $Q^{(M)}(x,t) \geq 0$, we have

$$\lim_{M \to \infty} Q^{(M)}(x,t) = Q(x,t)$$

exists and satisfies (24). This follows since we can pass to the limit under the integral.

Thus we have shown that the problem of calculating

$$\int_{S_t} e^{-\int_0^t V(x(s))ds} \, d_w x$$

is related to the problem of solving the integral equation (24). Now we shall show that (24) implies the differential equation

$$(27) \qquad \frac{\partial Q}{\partial t} = \frac{1}{4}\frac{\partial^2 Q}{\partial x^2} - V(x)Q.$$

We first make a change of variable in the inner integral. Let $z = \frac{y-x}{\sqrt{t-s}}$ or $y = \sqrt{t-s}\,z+x$. We obtain

$$(28) \qquad Q(x,t) + \int_0^t\int_{-\infty}^{\infty}\frac{e^{-z^2}}{\sqrt{\pi}} V(\sqrt{t-s}\ z+x)Q(\sqrt{t-s}\ z+x,s)dzds$$

$$= Q_0(x,t).$$

Now taking the partial with respect to t of both sides of (28), we obtain

$$(29) \qquad \frac{\partial Q}{\partial t} + \int_{-\infty}^{\infty} \frac{e^{-z^2}}{\sqrt{\pi}} V(x)Q(x,t)dz +$$

$$\int_0^t \int_{-\infty}^{\infty} \frac{e^{-z^2}}{\sqrt{\pi}} \frac{z}{2\sqrt{t-s}} \left(V'(\sqrt{t-s}\, z+x)Q(\sqrt{t-s}\, z+x,s) + \right.$$

$$\left. V(\sqrt{t-s}\, z+x)Q_{(1)}(\sqrt{t-s}\, z+x,s)\right)dzds$$

$$= \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}} \left(\frac{x^2}{t^2} - \frac{1}{2t}\right),$$

where prime indicates derivative with respect to the argument and subscript (1) indicates partial with respect to the first variable. Making the change of variable $y = \sqrt{t-s}\, z + x$ in (29), we obtain

$$(30) \qquad \frac{\partial Q}{\partial t} + V(x)Q(x,t) +$$

$$\int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} \frac{y-x}{2(t-s)} D_y(V(y)Q(y,s))dyds$$

$$= \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}} \left(\frac{x^2}{t^2} - \frac{1}{2t}\right).$$

Now we take the partial of both sides of (24) with respect to x. For the first partial we have

$$\frac{\partial Q}{\partial x} + \int_0^t \int_{-\infty}^{\infty} D_x \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y)Q(y,s)dyds = \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}} \left(\frac{-2x}{t}\right).$$

And for the second we have

$$(31) \qquad \frac{\partial^2 Q}{\partial x^2} + \int_0^t \int_{-\infty}^{\infty} D_x^2 \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} V(y)Q(y,s)dyds$$

$$= \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}} \left(\frac{4x^2}{t^2} - \frac{2}{t}\right).$$

Now if we consider the inner integral in (31) and integrate by parts once, we obtain

$$(32) \qquad \frac{\partial^2 Q}{\partial x^2} + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{\pi(t-s)}} \left(\frac{-2(x-y)}{t-s}\right) D_y V(y)Q(y,s)dyds$$

$$= \frac{e^{-\frac{x^2}{t}}}{\sqrt{\pi t}} \left(\frac{4x^2}{t^2} - \frac{2}{t}\right).$$

Thus adding -¼ of (32) to (30), we obtain

$$\frac{\partial Q}{\partial t} + V(x)Q - \frac{1}{4}\frac{\partial^2 Q}{\partial x^2} = 0$$

which of course is (27). By (25) and (26) we have the conditions

$$(33) \quad \lim_{|x|\to\infty} Q(x,t) = 0 \text{ and } \lim_{t\to 0}\int_{-\epsilon}^{\epsilon} Q(x,t)ds = 1 \text{ for } \epsilon > 0.$$

In the above, we differentiated under the integral since the continuity of the partials under the integral is guaranteed by the integral equation itself. Likewise we can use integration by parts since $V(y)\cdot Q(y,s)$ is absolutly continuous in y. [9].

The partial differential equation (27) can be handled by the method of separation of variables. We obtain the two equations

$$(34) \qquad \frac{1}{4}\,\Theta''(x) + (\lambda - V(x))\Theta(x) = 0$$

$$(35) \qquad \psi'(t) + \lambda\psi(t) = 0.$$

Equation (35) has the solution

$$(36) \qquad \psi(t) = ce^{-\lambda t}.$$

Before going further here, let us consider the case $V(x) = x^2$. Now (34) becomes

$$(37) \qquad \frac{1}{4}\,\Theta''(x) + (\lambda - x^2)\Theta(x) = 0.$$

Let $y = \sqrt{2}\,x$. Then we obtain

$$(38) \qquad \phi''(y) + (2\lambda - y^2)\phi(y) = 0$$

Let us assume a solution of the form $\phi(y) = e^{-\frac{y^2}{2}}u(y)$. Then from (38) we find

$$(39) \qquad u''(y) - 2yu'(y) + (2\lambda-1)u(y) = 0.$$

Now we assume that u is a power series in y. Then we have

$$(40) \qquad u(y) = a_o + a_1 y + a_2 y^2 + \dots + a_n y^n + \dots.$$

Differentiating (40) term by term we find

$$(41) \qquad u''(y) = 2a_2 + 3\cdot 2a_3 y + 4\cdot 3a_4 y^2 + \dots + n(n-1)a_n y^{n-2} + \dots$$

$$(42) \qquad -2yu'(y) = -2a_1 y - 2\cdot 2a_2 y^2 - \dots - 2na_n y^n - \dots$$

$$(43) \qquad (2\lambda-1)u(y) = (2\lambda-1)a_o + (2\lambda-1)a_1 y + \dots.$$

Since (41) plus (42) plus (43) is zero, the coefficient of y^n must be zero. Hence

$$(n+2)(n+1)a_{n+2} - 2na_n + (2\lambda-1)a_n = 0.$$

Thus

$$(44) \qquad \frac{a_{n+2}}{a_n} = \frac{2n-2\lambda+1}{(n+2)(n+1)}.$$

By (33) we see that the series (40) must terminate.
Hence

$$2n - 2\lambda + 1 = 0,$$

and

(45) $$\lambda = \frac{2n+1}{2}.$$

Relation (44) and (45) implies that $u = cH_n$ is the solution to (39), where H_n is the Hermite polynomial of degree n.

Thus the solution to (37) is

(46) $$\Theta(x) = ce^{-x^2} H_n(\sqrt{2}\, x) \quad \text{for } \lambda = \frac{2n+1}{2}.$$

Therefore, (27) has the solution

(47) $$Q(x,t) = \sum_{j=0}^{\infty} c_j e^{-\frac{2j+1}{2}t}\, e^{-x^2} H_j(\sqrt{2}\, x).$$

Condition (33) can now be applied,

$$\lim_{t \to 0} \int_{-\infty}^{\infty} Q(x,t) \cdot e^{-x^2} H_j(\sqrt{2}\, x)dx = H_j(0)$$

and

$$\sum_{i=0}^{\infty} c_i \int_{-\infty}^{\infty} e^{-2x^2} H_i(\sqrt{2}\, x) H_j(\sqrt{2}\, x)dx = \frac{1}{\sqrt{2}}\, 2^j\, j!\sqrt{\pi} \cdot c_j.$$

Therefore

$$(48)\qquad Q(x,t) = e^{-\frac{t}{2}}\sqrt{\frac{2}{\pi}}\sum_{i=0}^{\infty}\frac{e^{-\frac{1}{2}(2x^2)}}{2^i\, i!}(e^{-t})^j H_j(\sqrt{2}\,x)H_j(0).$$

Using a well known formula [10], we find

$$Q(x,t) = e^{-\frac{t}{2}}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{1-e^{-2t}}}\,e^{x^2-\frac{2x^2}{1-e^{-2t}}}.$$

We thus have

$$(49)\qquad Q(x,t) = e^{-\frac{t}{2}}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{1-e^{-2t}}}\,e^{-x^2\frac{1+e^{-2t}}{1-e^{-2t}}}.$$

Using (25) we find

$$(50)\qquad \int_{\Omega} e^{-\int_0^t x^2(s)ds}\, d_w x = \sqrt{2}\; e^{-\frac{t}{2}}\,\frac{1}{\sqrt{1+e^{-2t}}} = (\operatorname{sech} t)^{\frac{1}{2}}.$$

Returning now to the general case, we have from (26), since by (24) Q is continuous,

$$(51)\qquad \lim_{\epsilon\to 0}\frac{\int_{S_t} e^{-\int_0^t V(x(s))ds}\, d_w x}{\epsilon} = Q(a,t)$$

where $S_t = \{x : a \leq x(t) \leq a + \epsilon\}$.

Suppose now that $V(x) \to \infty$ as $x \to \pm\infty$. Under this assumption the differential equation

$$(52) \qquad \frac{1}{4}\,\theta''(x) - V(x)\theta(x) = -\lambda\,\theta(x)$$

is known to have discrete eigenvalues

$$\lambda_1 , \lambda_2 , \cdots$$

with corresponding eigenfunctions

$$\theta_1 , \theta_2 , \cdots .$$

For discussion of this problem see Titchmarsh [11].

It also follows from (33) and (36) that we can write

$$(53) \qquad Q(a,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\,\theta_j(a)\theta_j(0),$$

if the eigenfunctions are normalized.

Now it is clear that we could consider a space of functions $\underline{x}$ such that $\underline{x}(0) = \xi$. In fact this space of functions is equivalent to Ω since we can write $\underline{x}(t) = x(t) + \xi$. Thus we would have

$$(54)\qquad \lim_{\epsilon\to 0}\frac{\displaystyle\int_{S_t} e^{-\int_0^t V(\xi+x(s))ds}\,d_wx}{\epsilon} = \sum_{j=1} e^{-\lambda_j t}\theta_j(a)\theta_j(\xi)$$

where

$$S_t = \{\underline{x}:a\leq \underline{x}(t)\leq a+\epsilon\} = \{x:a-\xi\leq x(t)\leq a-\xi+\epsilon\} .$$

Now we will show that some of the properties of the classical expression

$$\sum_{j=1}^{\infty} e^{-\lambda_j t}\theta_j(a)\theta_j(\xi)$$

can be deduced from the Wiener integral. If, in (54), we let $\xi = a$ and note that as $t\to 0$

$$\int_{S_t} e^{-\int_0^t V(\xi+x(s))ds}\,d_wx \sim \underline{m}(S_t) = \underline{m}\{x:0\leq x(t)\leq\epsilon\}$$

then we see that

$$(55)\qquad \lim_{\epsilon\to 0}\frac{\underline{m}\{x:0\leq x(t)\leq\epsilon\}}{\epsilon} \sim \sum_{j=1}^{\infty} e^{-\lambda_j t}\theta_j^2(\xi) \text{ as } t\to 0.$$

But

$$\lim_{\epsilon \to 0} \frac{\underline{m}\{x : 0 \leq x(t) \leq \epsilon\}}{\epsilon} = \lim_{\epsilon \to 0} \frac{\int_0^{\epsilon} \frac{e^{-\frac{y^2}{t}}}{\sqrt{\pi t}}\, dy}{\epsilon}$$

$$= \frac{1}{\sqrt{\pi t}}$$

Hence we have (56)

$$(56) \qquad \sum_{j=1} e^{-\lambda_j t} \theta_j^2(\xi) \sim \frac{1}{\sqrt{\pi t}} \quad \text{as } t \to 0.$$

Now we use a classical tauberian theorem for Dirichlet series originally stated by Hardy and Littlewood in [12]. This theorem is given in a slightly extended form by D. Widder in [13]. This theorem states that if $\alpha(s)$ is non-decreasing and such that the integral

$$(57) \qquad f(t) = \int_0^{\infty} e^{-ts} d\,\alpha(s)$$

converges for $t > 0$, and if for some non-negative number γ and some A

$$(58) \qquad f(t) \sim \frac{A}{t^{\gamma}} \qquad \text{as } t \to 0+$$

then

$$\alpha(s) \sim \frac{A\, s^{\gamma}}{\Gamma(\gamma+1)} \qquad \text{as } s \to \infty . \tag{59}$$

Now letting $\alpha(s) = \sum_{\lambda_j < s} \theta_j^2(\xi)$, (57) reduces to

$$f(t) = \sum_{j=1} e^{-\lambda_j t}\, \theta_j^2(\xi). \tag{60}$$

Then (56) becomes (58) with $\gamma = \frac{1}{2}$, $A = \frac{1}{\sqrt{\pi}}$.
Hence we have from (59), since $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$

$$\sum_{\lambda_j < s} \theta_j^2(\xi) \sim \frac{2\sqrt{s}}{\pi}. \tag{61}$$

Thus we have shown that the eigenfunctions of (34) obey (61) for all continuous functions V(x). The probabilistic expression has allowed us to draw a purely classical conclusion.

CHAPTER IV

THE KOLMOGOROV-DOOB FORMULATION OF THE WIENER PROCESS

In this chapter, the intention is to indicate without details some of the important aspects of an alternative formulation of the Wiener process. This alternative formulation follows the general program proposed by Kolmogorov [5] in 1933. Many special aspects of this program have been studied by Doob, Loève, Lévy, and others, in the succeeding years.

A stochastic process $\{x_t\}$ is an indexed set of random variables which we shall here assume to be real-valued. We take the interval $[0,1]$ as the indexing set and may think of it as the time interval on which the process is considered. In order to speak of random variables, there must be a non-empty set Ω (the sample space) on which the random variables are defined, a σ-field $\mathcal{F}$ of subsets of Ω, and a probability measure p with domain $\mathcal{F}$. For each value of the index t, x_t is a real-finite-valued function with domain Ω. The function x_t is also assumed to be measurable with respect to $\mathcal{F}$. That is, if B is any Borel set of real numbers, the ω set $\{\omega : x_t(\omega) \in B\}$ belongs to the σ-field $\mathcal{F}$. Now if we think of fixing a point ω, we can consider

$x_t(\omega)$ as the value of a function of t at the point t. This function of t then is a "realization" of the stochastic process. That is, the function of t, $x_t(\omega)$, gives the outcome of the stochastic process for a particular $\omega \varepsilon \Omega$, at each $t \varepsilon [0,1]$ or at each instant of time. Letting ω range over some set in Ω, we have the notion of an ensemble or collection of realizations of the process.

This rather general formulation can be applied to many specific problems in which we can generally choose the space Ω, and the σ-field $\mathcal{F}$, in various ways. For a Brownian motion process, one such choice follows one of the methods proposed by Kolmogorov [5].

Suppose that Ω is the set of all real-valued functions vanishing at the origin and with domain $[0,1]$. Then each point ω is a function on $[0,1]$ whose value at ξ we will denote by $\omega(\xi)$. Now we let the random variables x_t be such that $x_t(\omega) = \omega(t)$.

Now to construct a σ-field $\mathcal{F}$, we first consider a finite set $\{t_1, t_2, \ldots t_m\}$ of values of t. Then we will specify that the ω set $\{\omega : x_{t_i}(\omega)\ B_i,\ i=1, \ldots, m\}$ shall be in $\mathcal{F}$. Now let us notice that these sets form a field $\underline{\mathcal{F}}$. The union of two sets in $\underline{\mathcal{F}}$ is again a set in $\underline{\mathcal{F}}$ and the complement of a set in $\underline{\mathcal{F}}$ is also in $\underline{\mathcal{F}}$. The whole space Ω is in $\underline{\mathcal{F}}$ and so is the empty set $\emptyset$. The minimal Borel extension of the field $\underline{\mathcal{F}}$ will be the

σ-field $\mathcal{F}$ in this case.

Now it is required to define a countably additive measure p of the sets F in $\underline{\mathcal{F}}$. This definition would have to be consistent in the sense that the measure assigned to any F in $\underline{\mathcal{F}}$ is independent of the representation of F. For example, F has representations of the forms $\{\omega : x_{t_i}(\omega) \in B_i\}$ and $\{\omega : x_{t_{\mu_i}}(\omega) \in B_{\mu_i}\}$, where $\mu_1, \ldots, \mu_m$ is some permutation of the integers from 1 to m. Also, we have a representation of the form $F = \{\omega : x_{t_i}(\omega) \in B_i, x_{t_{m+1}}(\omega) \in R_1, x_{t_{m+2}}(\omega) \in R_1\}$, where R_1 is the real line. The Einstein-Smoluchowski model has these consistency properties for the sets F in $\underline{\mathcal{F}}$. The specification of the measure of a set $F \in \underline{\mathcal{F}}$ requires that we arrange the t points in ascending order, guaranteeing the first consistency condition. The second consistency condition is stated by the formula (3) given in the Introduction.

Now Carathéodory's extension theorem enables us to extend the measure p on the field $\underline{\mathcal{F}}$ uniquely to the σ-field $\mathcal{F}$. This follows from the fundamental theorem of Kolmogorov in [5]. This then gives us a probability measure p on the σ-field $\mathcal{F}$ in the space Ω of functions vanishing at the origin. This measure agrees with the Wiener measure discussed in Chapter II in accordance with the uniqueness assertion of the Carathéodory extension.

Doob has shown [6] that the use of the space of <u>all</u> functions does not lead to a very satisfactory σ-field $\mathcal{F}$. The set of continuous functions contained in Ω for instance, turns out to be non-measurable. The set $\{\omega : x_t(\omega) \leq c$, for all t in $[0,t_1]\}$ is not measurable, whereas the set $\{\omega : x_t(\omega) \leq c$, for all rational t in $[0,t_1]\}$ is measurable. This state of affairs is highly undesirable in the study of a mathematical model of the Brownian movement.

One way of correcting this difficulty is to restrict the space Ω to the set of all <u>continuous</u> functions on $[0,1]$ which vanish at the origin. This is, of course, what we have done in Chapter II.

Having obtained a countably additive probability measure, the Wiener integral of a measurable function is simply the abstract Lebesgue integral as studied in the general theory of probability.

BIBLIOGRAPHY

LITERATURE CITED

1. Einstein, A., "Zur Theorie der Brownschen Bewegung," Annalen der Physik. Series IV, Vol. 19, (1906), pp. 371-381.

2. Smoluchowski, M. von, "Uber den Begriff des Zufalls und der Ursprung der Wahrscheinlichkeitsgesetze in der Physik," Die Naturwissenschaften. Vol. 6, (1918), pp. 253-263.

3. Bachelier, L., "Théorie de la Spéculation," Annales Scientifiques de l' École Normale Supérieure Paris. Series 3, (1900), pp. 21-86.

4. Wiener, N., "Generalized Harmonic Analysis," Acta Mathematica. Vol. 55, (1930), pp. 214-221.

5. Kolmogorov, A. N., Foundations of the Theory of Probability. Trans. N. Morrison, New York: Chelsea Publishing Company, (1950), p. 29.

6. Doob, J. L., "Stochastic Processes Depending on a Continuous Parameter," Transactions of the American Mathematical Society. Vol. 42, (1937), pp. 107-140.

7. Graves, L. M., The Theory of Functions of Real Variables. 2nd ed. New York: McGraw-Hill Book Company, Inc., (1956), pp. 117-118.

8. Kac, M., Probability and Related Topics in Physical Sciences. Seminar in Applied Mathematics, American Mathematical Society and University of Colorado. Boulder, Colorado, (1957), Chap. V.

9. Graves, op. cit., Chap. XI.

10. Titchmarsh, E. C., Introduction to the Theory of Fourier Integrals. 2nd ed. London: Oxford University Press, (1948), p. 77.

11. Titchmarsh, E. C., Eigenfunction Expansion Associated with Second-Order Differential Equations. London: Oxford University Press, (1946), Chap. 5.

12. Hardy, G. H., and Littlewood, J. E., "Some Theorems Concerning Dirichlet's Series," Messenger of Mathematics. Vol. 43, (1914), pp. 134-147.

13. Widder, D. V., The Laplace Transform. Princeton: Princeton University Press, (1946), Chap. V.

ADDITIONAL REFERENCES

1. Bachelier, L., Calcul des Probabilites. Gauthier-Villars, Paris, (1912), Chap. 16.

2. Cameron, R. H. and Martin, W. T., "An Expression for the Solution of a Class of Non-linear Integral Equations," American Journal of Mathematics. Vol. 66, (1944), pp. 281-298.

3. Cameron, R. H. and Martin, W. T., "Transformations of Wiener Integrals under Translations," Annals of Mathematics. Second Series, Vol. 45, (1944), pp. 386-396.

4. Cameron, R. H. and Martin, W. T., "Transformations of Wiener Integrals under a General Class of Linear Transformations," Transactions of the American Mathematical Society. Vol. 58, (1945), pp. 184-219.

5. Kac, M., "On Some Connections between Probability Theory and Differential and Integral Equations," Proceeding of the Second Berkeley Symposium on Mathematical Statistics and Probability. ed. J. Neyman. Berkeley: University of California Press, (1950), p. 189.

6. Kac, M., "Distribution of Eigenvalues of Certain Integral Operators," Michigan Mathematical Journal. Vol. 3, (1955-1956), pp. 141-148.

7. Levy, P., Processus Stochastiques et Mouvement Brownien. Paris: Ganthier-Villars, (1948).

8. Paley, R. E. A. C. and Wiener, N., "Fourier Transforms in the Complex Domain," American Mathematical Society Colloquium Publications. Vol. 19, (1934), Chap. 9.

9. Skitovic, V. P., "On Characterizing Brownian Motion," Teoriya Veroyatnoster i ee Primeneniya, Akademiya Nank SSSR. Vol. 1, (1956), pp. 361-364. (Russian. English Summary).

10. Tulcea, C. T. Ionescu, "Mesures dans les Espaces Prodiuts," Atti della Accademia Nazionale dei Lincei, Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Series 8, Vol. 7, (1949), pp. 208-211.

11. Wiener, N., "Differential Space," Journal of Mathematics and Physics. Vol. 2, (1923), pp. 131-174.

12. Wiener, N., "The Homogeneous Chaos," American Journal of Mathematics. Vol. 60, (1938), pp. 897-936.